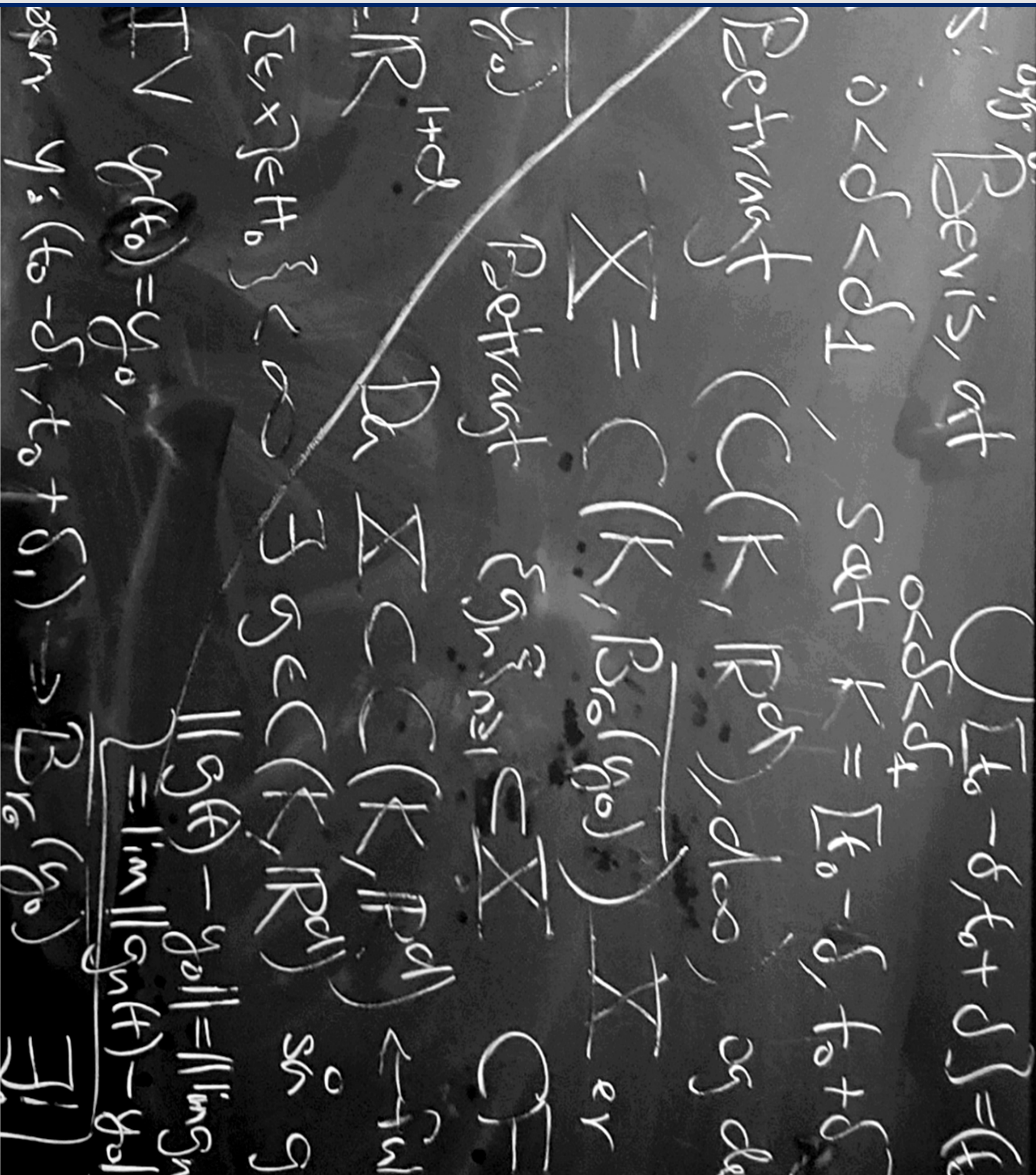
# The Aalborg Mathematician

September 2024



## Founder's message

I had the privilege and honor last year to bring together a number of students to form the first AAU Mathematical Sciences Society committee: Jens Peter Østergaard Knudsen (President), Magnus Bredvig (Vice President), Marcus Basse Pedersen (Treasurer), and Viktor Ernst Dahl Rasmussen (Editor). They have acquitted themselves admirably. Thanks to them, the Department of Mathematical Sciences is able to boast the first alumni reunion dinner that was held in May this year. Hopefully, this is the first of many to come!

From the moment that students arrive at university, they are conditioned for exams that are the culmination of every semester. This has proven to be an efficient method of teaching and learning. But a student's university journey is so much more, there are the lasting friendships that are formed, the fun that is had, and the pursuit of a future career. The Society aims to serve the student body by enhancing all aspects of this journey.

The Society is incredibly grateful to Orient's Fond. It would be no exaggeration to say that without their financial support, the Society may well have not come into existence.

*Costy Kodsi*

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## Editor's message

The inevitable fusion of mathematical sciences, as the ultimate manifestation of objective exactness, presented through a subjective and therefore imprecise selection of words presents a wholly underestimated challenge in my opinion. I, therefore, appreciate how this first issue of *The Aalborg Mathematician* aims to present its readers with an introductory view through the keyhole to the sphere of cultures and contexts that surround the field and the department at Aalborg University: Spanning from light-hearted articles that describe our assistant professors' profiles to a student's insight into what they describe as 'Real Math'. Though my inexperience as the editor at the Aalborg University Mathematical Sciences Society is sure to permeate throughout this issue, I have wholeheartedly enjoyed humbly working with each and every contributor and their distinguished take on the aforementioned challenge.

*Viktor Ernst Dahl Rasmussen*

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# The Life of a Ph.D. Student

*Olivia My Töffner Kvist and Viktor Rasmussen*

Olivia My Töffner Kvist

The following article is the transcript of an interview with Ph.D. student Olivia My Töffner Kvist conducted on the 11 of April 2024 by Viktor Rasmussen at the Department of Mathematical Sciences at Aalborg University.

Viktor Rasmussen: Olivia, would you please introduce yourself?

Olivia My Töffner Kvist: I was born and raised in Copenhagen. When I had to apply for university, my original plan was to study HA(mat.) at CBS (Copenhagen Business School), but I didn't get in. I then looked online to see what free spots were available: Aalborg University came up. I had heard plenty of positive things from AAU and from several different people. Luckily, the Mathematics-Economics program had spots available, so I ended up applying there.

VR: So were these positives linked to the PBL (problem-based learning) model at AAU?

OMTK: Yes - mostly. The PBL model at AAU and the group work involved make you more inclined to make compromises, tune into what you and other people want, and make those align. Also, the Department of Mathematical Sciences, as we know, is quite small and is, therefore, quite intimate regarding how the students get to know the lecturers and vice versa. I don't necessarily think that you see this in bigger departments as there are so many students and lecturers.

VR: Regarding your Ph.D., could you introduce your topic?

OMTK: The main theme of my Ph.D. is climate econometrics, which is just a fancy word for statistical and mathematical models used for economic data and problems. My Ph.D. is specifically focused on climate, and so I am trying to figure out whether there is a correlation between climate change and the risk it implies within the financial sector and the other way around; how they may affect each other. As of now, I have done two papers overall, both of which are waiting to be published.

VR: I do suppose that Dansk Landbrug is somewhat of a hot subject within this context at the moment?

OMTK: It really is, but it is not something that I am particularly interested in. My approach would be more like this: I can see this tendency and this trend concerning how climate change is affecting these kinds of stocks or businesses, and so how can the government go in and make policies such that the risk regarding climate change is minimized.

VR: How did you come to choose this topic for your Ph.D.?

OMTK: The topic was already set in place by the department that finances my Ph.D. So the department had a meeting where staff members with ideas for Ph.D. topics discussed which topic should be financed. Ultimately, that choice fell on J. Eduardo Vera-Valdés.

VR: In terms of your personal motivations for writing a Ph.D., do these rely on the specific topic or simply being able to do research?

OMTK: Both. If the topic did not speak to me, I would have never applied for this Ph.D., and I don't think you can actually do a Ph.D. in a topic in which you are not invested: It is too much work and too intense, and so you have to be interested in the topic to succeed. My motivations for doing the Ph.D. were of course based on the topic of the Ph.D., but I also do actually like studying, and I was unsure of what to do after my master's degree. I was fortunate to be provided with the opportunity to write this Ph.D. and found it a way to get even more specialized within a topic that I was interested in. Furthermore, if you like to study and do research, writing a Ph.D. is a great opportunity. I am also considering staying in academia, and if you want to do that, then it is a really good idea to have a Ph.D.

VR: Do you have any personal ambitions regarding this Ph.D.?

OMTK: I am not entirely sure whether I want to explore my opportunities in the private sector or stay in academia yet. As of right now, I am leaning towards staying in academia and preferably here in the department; I am really settling in here.

VR: I gather that doing a Ph.D. can be rather intense. Do you have any specific scheduling you would like to share?

OMTK: For my semesters till now, I have had some teaching obligations, or, rather, I was asked to teach topics related to financial markets. I naturally had some scheduling to do teaching-wise, which had to fit in with the program. Furthermore, I work with a lot of deadlines, e.g. we have this yearly conference: Econometrics Models of Climate Change, which is *the* conference that is interesting for me to participate in. Right now, I am working on an extended abstract to submit to that conference, which will hopefully provide me with the opportunity to speak at this conference. Overall, my scheduling is really free and it depends on how you work best: Most of the time, if I don't have anything at the department, and because the offices are then empty, I mostly work from home and do it at my own pace.

VR: So it could be midnight, midday, etc.?

OMTK: Yes - depending on what works best. I do not really work from 8:00-16:00, I just work whenever it is needed and when I have something to do, which is basically all of the time.

VR: In terms of applying for a Ph.D., are there any skills you hold in particularly high regard?

OMTK: It is definitely not for everyone. I think that time management is a big necessity: You must be able to prioritize which project you must be working on, and I also think that a lot of Ph.D. students are sort of perfectionists concerning what they are working on. This is good, as it ensures the quality of their work, but it can also hold you back from finishing a paper. Similar to the semesters' projects, you can always keep adding to them, but at some point, you have to stop. You can't spend your whole Ph.D. on a single paper: You must focus on at least three papers throughout your Ph.D., and so you must set some limits as to when you are done with a specific paper. Sure I can keep adding things, but at some point, you must tell yourself that this is it.

VR: Have you had problems balancing your life outside the Ph.D. with your specific approach to time management?

OMTK: I haven't found issues with it, but since I have spent six years in this department, I have found that my life at the university and my life outside have fused. You can run into trouble, which is why you have to prioritize time-management skills and allow yourself to take some time off, which ties in with how the perfectionist approach may affect some students negatively.

VR: Have there thus far been any hiccups in your Ph.D.?

OMTK: The most difficult part of your Ph.D. is whether your papers get published or not. I had a paper that was submitted in late January (2024) to a journal and got rejected: I did not get any feedback because it did not make it to the review process as we applied to a special issue of the journal, and the paper fell outside the scope of this special issue. Considering what the normal edition of the journal includes, I think that my paper totally falls within its scope. So, yeah, I think that the most difficult part is being able to rejection in this regard.

VR: So we are nearing the end of this interview, but I wonder whether there are any reasons why you chose to do your Ph.D. at AAU?

OMTK: It was a natural extension of my master's as my Ph.D. is integrated, and thus, I am not really a master's yet. I also think that the PBL model and project-focused work provided me with knowledge and experience with working with the professors at the department, which meant that I was familiar with how we would work together. Otherwise, I might have gone to work with a complete stranger and wouldn't know whether our ambitions or work ethics would correlate.

# Our Assistant Professors' Profiles

*Fynn J. Aschmoneit, Francesco Benvenuti, Matteo Bonini, René B. Christensen, Charisios Grivas, Aysegül Kivilcim, Costy Kodsi, Vytaute Pilipauskaite, and Søren B. Vilsen*

This article serves as an introduction to the assistant professors of the Department of Mathematical Sciences at Aalborg University. Each assistant professor has been asked to provide answers to the following list of descriptive questions:
**1.** *Favorite drink?*
**2.** *What do you teach?*
**3.** *Do you have a teaching philosophy?*
**4.** *What is your research about?*
**5.** *What are your typical working hours?*
**6.** *What piece of advice do you wish you had when you started your degree?*
**7.** *If not a mathematics-based degree, what would you have liked to study?*
Note that not all professors provided answers for every question.

**Fynn Jerome Aschmoneit** from Flensburg, Germany. **2.** Fundamental mathematics in various engineering programs. **4.** Fluid mechanics and microfluidic multiphase flows.

**Francesco Benvenuti** from Venice, Italy. **1.** Espresso. **2.** Econometrics and microeconomics. **3.** I primarily have two goals related to my teachings: First, to make students interested in the subject such that they may invest more time in dwelling on it at home, and second, to guide them in their understanding of the ideas behind a formula or proof, since intuition is really important in mathematics. The practical application of those, of course, depends on the class, their respective levels, etc. **4.** Financial econometrics. In particular, high-frequency econometrics. **6.** To try to obtain an overview of all the different topics related to economics and not just the subjects that I was taking, in order to understand how broad the subject of economics can be: From law to stochastic processes, there are many possibilities to pursue according to your inclinations when you study within this field.

**Matteo Bonini** from Perugia, Italy. **1.** Beer. **2.** Algebra 1 and 2, and I supervise most of the projects in discrete mathematics. **3.** In my teaching philosophy, I prioritize explaining the fundamental ideas behind mathematical concepts and introducing them rigorously. I believe in a balance between conceptual understanding and problem-solving skills, and I aim to inspire curiosity and promote analytical thinking. **4.** My research focuses on coding theory and cryptography. In particular, I look at communication processes through the lens of algebraic and geometric structures, with the goal of improving the reliability of communications, both in terms of error correction and security from malicious attacks. **6.** Study more :). **7.** Probably chemistry.

**René Bødker Christensen** from Løgstør, Denmark. **1.** Interpreting 'drink' as 'cocktail', I would probably highlight the cheesily named 'Between the Sheets': 4 parts cognac, 2 parts Cointreau, 2 parts Bénédictine, 1 part lemon juice. Shaken with ice and served 'up'. **2.** Primarily linear algebra in the first year, but I have recently taken over half of the LAMA (linear algebra with applications) course in the third semester. Last summer I was also involved in a Ph.D. course on quantum computing. **3.** Calling it a philosophy is probably exaggerated, but I try to make subjects more accessible, i.e. I try to not let small details muddy the overall idea and intuition. Moreover, I try to facilitate the validity of so-called 'stupid questions'. They are only 'stupid' if you already have a good insight and understanding. If not, asking those questions is a perfectly valid way to reach such an understanding. **4.** Very briefly, it is about using the unique properties of quantum information to solve problems in communication theory and cryptography. **5.** That depends a lot on the workload in a given week. In general, I prefer to show up early rather than staying late. **7.** Realistically, it would probably have been something closely related to my degree, i.e. computer science. If we leave realism (and reason) out of the question, it would be cool to be a cobbler (or maybe a cooper to match my middle name).

**Charisios Grivas** from Kozani, Greece. **1.** Coffee. **2.** Time-series econometrics. **3.** I try to foster an environment in which students feel free to ask questions without fearing that they will make mistakes. In my opinion, this allows students to feel relaxed and ask questions about topics they find interesting. To support this environment, I also produce my own teaching materials, which are often based on questions I gather from students during the lectures. **4.** Theoretical econometrics, financial econometrics, and macroeconometrics. I am an econometrician who employs tools from econometrics and statistics to contribute to econometric theory and other fields of statistics. **7.** Perhaps law or politics.

**Aysegül Kivilcim** from Balikesir, Turkey. **1.** Coffee. **2.** Ordinary differential equations, calculus, linear algebra, and applied mathematics for engineers. **3.** My teaching philosophy revolves around improving the students' involvement in the lectures and reaching each individual student in these classes. I also believe that learning best revolves around asking proper questions, not being afraid of making mistakes, and teaching the subjects that you are involved with to your peers. **4.** Stability of hybrid systems (impulsive differential equations and switched systems) and nonlinear control. **7.** Psychology or international relations.

**Costy Kodsi** from Raynes Park, London, U.K. **1.** Tea (three times a day) / Caol Ila (whisky) for them extra cold days. **2.** Differential geometry, and I supervise projects in differential equations. **3.** I cater my teaching to the abilities in a classroom. Generally, I like to mix theory and exercises in order for students to appreciate what they are taught. Most importantly, I want students to feel comfortable with me, so that they may ask questions and engage in constructive discussions. **4.** I mainly work in the application of mathematics in classical physics. More recently, however, I have turned my attention to data science. **5.** I do not really have any. There is a flexibility within academia that may not be found elsewhere. Unfortunately, this often translates into periods of very long hours being the mainstay. **6.** Whether you are tired, not in the mood or feeling down, just do any amount of work you can that day and keep on going every day! **7.** I can say with some certainty that I would have done history!

**Vytaute Pilipauskaite** from Pakruojis, Lithuania. **1.** Black coffee. **2.** Stochastic analysis, and I also do project supervision. **3.** I aim to introduce the main ideas broadly. A full understanding of a subject comes through working on the details of exercises. I always value and encourage the students' questions. **4.** Statistical inference for stochastic differential equations and limit theorems for stochastic processes with long-range dependence. **5.** I am most productive in the evening. **6.** Emphasize your effort in the core courses of your education and make the most of all the opportunities that the university offers. **7.** Economics.

**Søren Byg Vilsen** from Styding, a small town in southern Jutland. **1.** Coffee. **2.** Mainly applied statistics, machine learning, data mining, data science, and things of this nature. **4.** Applying statistics, machine learning, data mining, and so on, to different fields; specifically with a focus on battery health degradation and forensic genetics. **5.** Usually the standard 8-16 (or 9-17) plus an hour or two in the evenings. **6.** Focus on understanding the material, rather than passing the class. **7.** Probably history: I always enjoyed the process of understanding the point-of-view from which a text was written.

# About Ege

*Ege Rubak*

## The Early Years

I grew up in the town of Nibe, just 20 km north of Aalborg, Denmark. In the local elementary school, I was an all-round good student without being exceptional in any particular subject. I did not study hard and enjoyed hanging out with friends and playing all kinds of sports such as football, handball, volleyball, basketball, and golf. It was not until high school that I developed a strong interest in mathematics, and unlike most of my classmates, my grades improved compared to my years in elementary school. I really enjoyed the stimulating academic atmosphere in high school, where the teachers clearly had a talent and passion for mathematics and could challenge students at different levels. Luckily, mathematics came to me very easily and I was able to learn the curriculum without sacrificing a great deal of my time outside of high school. It was very important for me to have time to socialize, make new friends, and attend what seemed to be a never-ending stream of parties that all appeared to be the most important event of the year in a young teenager's mind. After my first year of high school, I had the opportunity to participate in a student exchange scheme, which took me to the U.S. I lived in Topeka, which is the capital of Kansas and of a similar size to Aalborg. I really enjoyed the stay, and it was really interesting to experience life in a typical middle-class American family where I learned a lot about U.S. culture and society. After graduating from the U.S. high school system, I returned to Denmark and finished the last two years of Danish high school. My core electives were mathematics and physics, and I knew that I wanted to study something related to those subjects. In these years, I often helped my classmates with their homework and I found this first experience with teaching mathematics very rewarding. When you have explained a concept from different angles, it is wonderful to see how your classmate all of a sudden 'just gets it'; I still enjoy that feeling as a teacher today.

## Gap Year and Undergraduate Degree

At first, I wanted to take a year off to work and travel after high school, but I never got around to actually applying for a job! In late August, I found that I could just enroll as an electrical engineering student at Aalborg University as there were spaces available. After the first semester, I realized that I was mostly interested in mathematics and physics, and not so much in electrical engineering. There was a joint 'basis' year for all educations in technical and natural sciences (including the mathematicians), which made it easy for me to change my choice of education, which led to my second-semester project in mathematics with Steffen Lauritzen (who would leave the department a few years later to take up a position at Oxford) as my supervisor. The project was about Bayesian networks in simple DNA cases of parenthood testing, and it contained exactly the right mix of theory and application that made it very interesting to me.

After that, I took a year off from studying to work and travel, which I actually followed through with. I have never regretted the decision to delay my studies for a year: The rather trivial manual labor jobs I did, including cleaning at a sausage factory, packing orders in a fireworks warehouse, and various other small jobs, and traveling through Central America for six months provided me with a lot of experiences and perspectives that I would not have had otherwise. When I returned to Aalborg University in September of 2003, it was my first real encounter with the Department of Mathematical Sciences, as my first year had been in the city center and detached from the individual departments within the natural sciences. The department was much smaller than my experience in my 'basis' year, with only one study program and approximately 15 students starting their third semester. The default option was to combine mathematics with another subject to become a high school teacher, but I really did not want to start a new subject from scratch, so I designed an individual study plan where I got a major in mathematics and a minor in communications technology, which involved taking a selection of courses at the Department of Electronic Systems. That meant that I could continue to write projects that were primarily mathematical while having the motivation and application coming from communications technology. For a long time, I was inclined to focus on mathematical analysis, since statistics seemed very strange to me, but on the other hand, it had many interesting applications. Ultimately, I ended up writing my thesis in probability/statistics with Martin Bøgsted, who is now heading AAU's Center for Clinical Data Science (CLINDA).

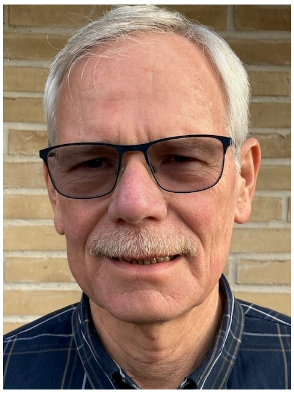
### Academia Calls


Ege Rubik, Adrian Baddeley, Rolf Turner

During my last year of studies, Jesper Møller got money from an international Ph.D.-student program. The idea was that he would hire a Ph.D. student who would spend half of their studies with some of his collaborators abroad, and he encouraged me to apply. I got the position and spent ten months in 2008 visiting Peter McCullagh at the University of Chicago, and then ten months in 2009 visiting Adrian Baddeley at the University of Western Australia in Perth. Both of these visits were extremely rewarding for me and shaped my academic profile. At the same time, I also got married to my wife Julie in 2008 and she joined me on both trips. In Chicago, I came to a much larger department with a dozen Ph.D. students in statistics and a lot of courses and seminars that I could attend. The other Ph.D. students were very friendly and welcoming, and very good at integrating me into the social life of the department. At the same time, it was an incredibly exciting time to be in Chicago as the local politician, namely one Barack Obama, who lived a few blocks from us, went from being a relatively unknown state politician to the president of the U.S. I will always remember the election night in Grant Park, where hundreds of thousands were gathered to celebrate. In Perth, the department was much smaller and there were not many other Ph.D. students to hang out with. However, I became a member of both the university volleyball club and scuba diving club, which allowed me to take up an old interest from my trip to Central America: Scuba diving in Western Australia is highly recommended! Academically, I developed a close collaboration with Adrian Baddeley, which ended up being career defining for me. Some years later he invited me to collaborate on an R software package called spatstat, which he had developed together with Rolf Turner from the University of Auckland. We then spent an insane amount of time writing a book about the analysis of spatial point patterns with spatstat, which was published in 2016, and it is the most cited scientific work that I have been involved in by a very large margin. The majority of the book was written in 2014 and 2015, which was a very intense period for me as my twin daughters were 2-3 years old, which made it a real challenge to balance my personal and work life. In the years after finishing my Ph.D. in 2010, I was employed first as a postdoc and then as an assistant professor at the department. Finally, in 2015, I was promoted to my current position as associate professor. In 2017, I took six months leave from my position to visit Adrian Baddeley again and continue our research collaboration funded by some grant money of his. This was a great experience as a family; to try something fundamentally different and live in a different country for a while. With time, I have been involved in more and more administrative tasks, and in 2023, I became head of the section of Statistics and Mathematical Economics.

# Interview with Martin Raussen

*Martin Raussen and Christian F. Skau*



Martin Raussen

## Introduction

This interview is replicated under the Creative Commons license 4.0 with permission from its participants. The interview was originally published in the European Mathematical Society Magazine no. 131 (2024), pp. 11-21.

Christian Skau (Norwegian University of Science and Technology, Trondheim, Norway) and Martin Raussen (Aalborg University, Denmark) teamed up to interview the Abel Prize recipients, from the first one in 2003 and until 2016, when Bjørn Dundas (University of Bergen, Norway) took over from Martin Raussen. Just before the opening of the 29th Nordic Congress of Mathematicians, organized in cooperation with the European Mathematical Society at Aalborg, Denmark, a session took place with Christian interviewing Martin. The text below is a slightly edited version of the recorded interview.

## Nordic Congress of Mathematicians

Christian F. Skau: Martin, the two of us have participated in many interviews together. This time there is nobody else to ask questions but me! We have a double special occasion: This afternoon the 29th Nordic Congress of Mathematicians, on the occasion of the 150 years anniversary of the Danish Mathematical Society, will get started here in Aalborg, and after the summer you will retire from your position at Aalborg University. Could you tell us about your involvement with the Nordic Congress?

Martin Raussen: I will try to do that, but before, let me thank you for coming at this special occasion. I am very grateful that you took the effort to come here from Oslo, where the two of us, for several years, have interviewed the Abel Prize recipients.

Before I answer your questions: Interviewing the Abel Prize recipients was like mingling with Champions League winners in soccer. I am not at all in that league. I would have difficulties to qualify for the Conference League! Therefore, I am very honoured to be granted this opportunity to be interviewed.

I started to work on getting the Nordic Congress to Aalborg around five years ago. In the beginning it took only little time, writing letters and applications. Forming a good and reliable team for the organization was essential. During the last year tasks accelerated. For the last month we have been very busy, indeed. But now we are happy that we can host more than 400 congress participants. There will be eight plenary talks at the House of Music, and we will have 29 special sessions in a university building very close by. The only thing that could be better, in my view, is the weather that has deteriorated quite a lot lately.

## First Mathematical Experiences

CFS: Let's now talk a little about you as a mathematician. My first question is: What kindled your interest and fascination with mathematics in the first place?

MR: My late father was a high school teacher in mathematics and physics. He was very interested in the subject and talked to me, and also to my brothers and sisters, about mathematical topics. He tried to give us small challenges from time to time. The first problem I still remember came when I was still a small kid, maybe just started school, or maybe second grade. He asked me, in a concrete setting dealing with brown and white hens, a question that, in hindsight, can be phrased as two simple linear equations in two variables. I had no clue about how to tackle this problem, but the numbers were not that large. I could just experiment in my head, and I found the answer after a while. He commented that this was somewhat exceptional for a kid at that age, and I was proud.

The second instance that still is in my memory happened when I was a bit older. I had heard about the test to decide whether a number is divisible by three or by nine by just taking the sum of the digits. And I started to think: How about other division tests of that kind? I would never have been able to find out about division by seven at that time! But by experimenting just with two- or three-digit numbers, and those were the numbers that were in my mind at that time, I found out with 11 it's the alternate sum that does the job. It got me excited! I remember that I wrote a small note about my finding and distributed it to some of my friends in the school class. They were not really interested in it, but it was my first experience of mathematical success.

CFS: That's a nice story! And a natural follow-up question: When did you discover that you really had special talent for mathematics?

MR: Talent and interest are very much related. I was a little bit older, still under the influence of my father, who at that time got interested in Boolean algebra and related concepts with the advent of the first digital computers. He led a small workshop on Boolean algebra. I participated, and I got the impression that as soon as you have something that you can formulate in terms of Boolean algebra, then the computer can work for you.

I started to look at what happens when you take as operations the greatest common divisor and the smallest common multiple of natural numbers, respectively. This gives you a distributive lattice, but in most cases the divisors of a natural number do not form a Boolean algebra. It works only well for square-free numbers. In hindsight, this is clear, because then the lattice corresponds to the power set of the prime divisors, but I did not know that at the time. Anyway, I wrote a small note and even tried to get it published in a small educational journal. It didn't work out, but still, I remember that I could tell myself: "I can somehow contribute to something that resembles a little bit of research." All that happened some time before I started to study at the university.

## A Mathematical Education

CFS: Very nice. Then let's move to your mathematical training. Tell us about your mathematical education, starting with the Max-Planck-Gymnasium in Trier, and until you got your Ph.D. in mathematics at Georg August University in Göttingen.

MR: That covers a range of years! I went to what's called Max-Planck-Gymnasium in Trier, in South West Germany. When people ask me about Trier, I answer: That is the city where Karl Marx was born! Apart from myself and many other people of course. The Max-Planck-Gymnasium, you can guess it from the name, was a high school where mathematics and natural sciences had a high level of esteem.

The other high schools in town either focused on old languages, Latin and Greek, or modern languages like French, which was important in Trier, being close to the borders of France and Luxembourg. At the Max-Planck-Gymnasium you were able to specialize in mathematical and physical directions, mainly during the last two or three years.

That gave me sort of a head start when starting at the University of the Saarland in Saarbrücken, around 100 kilometres south of Trier. I was only 17 years old then, and I started on an education in both mathematics and computer science, a subject that had just started a year earlier. In mathematics, we had to follow the standard curriculum: linear algebra and analysis. At first, I felt more at home in linear algebra because I had a rather algebraic mindset. I also liked to learn about Turing machines, computability, and formal languages in computer science.

Our analysis course consisted of three consecutive semesters that were given by a young professor at that time, Tammo tom Dieck, who had just got a professorship in Saarbrücken. He was a student of the famous topologist Dieter Puppe from Heidelberg. Professor tom Dieck did a very good job teaching us analysis from the elementary beginnings and ending with vector analysis. It was quite complicated stuff in the third semester. Both tom Dieck's personality, but also the drive of the group of young assistants surrounding him, his team, attracted me. It was the personalities of these people rather than the subject itself that impressed me most.

In retrospect, I am grateful for the inspiration they gave me. As a follow-up, there were offered courses in differential topology, which was developing rapidly at the time. After three further semesters we got close to the results on smooth structures on spheres by Kervaire and Milnor. Milnor became later one of our Abel interviewees! I felt a real challenge and excitement: After the first three years at university, we were at a level where we could read and understand, to a certain degree at least, papers that had been written perhaps five or ten years earlier. There were also interesting seminars where we were challenged to give talks ourselves about papers in the literature.

I should add that I received a grant from Studienstiftung des deutschen Volkes, a grant institution for talented university students, with money that made me financially independent. Since I also earned some money as an instructor for new students, I felt quite well-off at the time! The Studienstiftung also gave opportunities to attend summer academies during the vacations at nice places, in the Alps usually. That was challenging! I got to know other eager and talented students. Moreover, interesting subjects, not only within mathematics, were taught and discussed.

All that brought me up to a certain level. I asked for a topic for a master's thesis, and I got a suggestion by tom Dieck. It took some time to get the right mindset; the topic was no longer in differential topology, but in algebraic topology. I had never taken a class in algebraic topology, but during seminars, I had acquired some knowledge. I finished this master thesis within perhaps a year, finally typing it myself on an old-fashioned typewriter and inserting special letters by hand.

But then something else happened: My teacher tom Dieck was called to a professorship in Göttingen. Consequently, the whole team, his assistants and several of his students, followed him to Göttingen. My master's degree was obtained at the University of the Saarland, but the final oral exam took already place in Göttingen.

**Göttingen**

Saarbrücken and Göttingen are quite different. Göttingen has a lot of tradition: You can still walk to the observatory where Gauss worked, and the hall you enter at the department of mathematics is called the Hilbert space. You can walk along the offices where so many illustrious mathematicians had worked. The buildings of the department of mathematics, and that of physics nearby, had been built with support from the Rockefeller Foundation after the First World War.

CFS: And they were not destroyed during the Second World War?

MR: Göttingen was almost intact; only a few buildings were damaged. The city had not had any military importance; not that this fact helped many other German towns...

Göttingen is a relatively small city; you can get almost everywhere by walking. The university is old, and many facilities were old-fashioned. I had to learn to appreciate the charm. The University of the Saarland, on the other hand, was founded after the Second World War, with modern buildings and equipment.

Department of Mathematics, Göttingen University

CFS: How many students, approximately, were there at Göttingen?

MR: Göttingen is a relatively small town with a little more than 100,000 inhabitants at that time. I think there were around 20,000 students, probably 30,000 by now. The students really were, and still are, a very dominating section of the population. I don't know the exact numbers, but there were several hundred students of mathematics and physics.

CFS: From what you're telling me, it would seem to be natural that tom Dieck would be your Ph.D. advisor?

MR: In fact, I started to work under him. But then Larry Smith, a homotopy theorist at the time, was hired as a new young professor at the department. Every professor had two or three assistants, and I was offered one of these assistant positions. While I was still working on the first bits and pieces of my Ph.D. thesis, I was "taken over", so to speak, by

Larry Smith, and I was his assistant for several years. And then it seemed natural to say: Since you must work with this professor, he should also be your advisor. The transition went in fact quite easy.

Also, the topic of my work shifted a bit, quite naturally so. When you work on a thesis, you start an investigation into terra incognita. Some premature ideas do not work out, and then you try something else.

In the end, my thesis dealt with the homotopy classification of liftings into fibre bundles, and more concretely, with immersions and embeddings of smooth manifolds.

CFS: I see. You said earlier that you started being more interested in algebra, and then you switched to algebraic topology. . .

MR: In my view, topology is an ideal combination of several subjects. You can't do interesting topology without knowing some algebra, some analysis, and some geometry. In the beginning I didn't think I was very good at geometrical thinking at all. I've always been very bad at drawing. I've taught myself to do drawings relevant for mathematics, but apart from that, I have no artistic talent at all, I am afraid.

I have given a course in elementary differential geometry many times later in my teaching career. I then try to advertise the subject by telling the students that you can get to exciting results by stealing methods from analysis and linear algebra, combining them in a clever way. That synthesis has fascinated me, and I think that it made me like geometry and topology more than other mathematical areas.

## Mathematical Research

CFS: That's a nice description. Now, let us talk about your research after your Ph.D. Much of your recent research work has been about the interplay between what is called directed homotopy in algebraic topology and concurrency, a notion appearing in theoretical computer science. Could you please explain?

MR: Let me tell you about the next steps in my career first. As a Ph.D. student, I had the opportunity to study a year in Paris. While staying at the Cité universitaire, I got to know a Danish girl who later became my wife. Consequently, I had to learn Danish, and I moved from Göttingen to Denmark. I had short term positions at the Technical University in Lyngby, close to Copenhagen with Vagn Lundsgaard Hansen, and then in Aarhus with Ib Madsen, the grand old man in Danish topology. From there I applied, and finally got, a position as associate professor at Aalborg University Center – as it was then called – in the North of Jutland, the Danish peninsula.

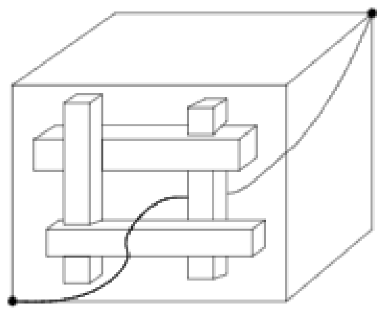
When I arrived, I was the only topologist at the department. I went regularly to seminars at Aarhus University and considered that as my research lifeline. I had to teach a lot, three small kids required attention, and time for research – at that time in equivariant algebraic topology – was scarce. A few years later, I was joined at the department by my good colleague and friend Lisbeth Fajstrup, a student of Ib Madsen.

Still, we felt a bit lonely research-wise and, encouraged by colleagues, we started to think about investigating applied areas of topology. An opportunity, a wild card, arose when we heard about a conference on New Connections between Mathematics and Computer Science at the Isaac Newton Institute in Cambridge, UK. We applied and were accepted. There were many interesting talks given during that week. Among them were two lectures that thrilled me from the very beginning: One outlining connections between what's called distributed computing and topology, and the other about connections between concurrency and topology; as it turns out, both are related to each other.

Let me tell you about concurrency and topology: It becomes interesting when you don't have just one program running on one processor, but when things get distributed, the latter being more and more the case. You want to do a calculation, or to run through an algorithm, and you distribute it either on various entities on your own laptop, or perhaps even on the entire World Wide Web. Anyway, there are several units that collaborate to arrive at a solution. In the extreme case, there is no coordination or very little coordination among them. In order to rely on a result of such a common effort with very little coordination, you need to have algorithms that are robust in the following sense: It should not play a role whether one processor achieves its goal very quickly, whereas another one is slowed down for some reason. If they communicate through some common registers, the order of access may be crucial, on the other hand.

For an easy case, consider

three processors, each working on a linear code without loops and branches. You can then interpret the compound process as a path in 3-dimensional space. But not every path can occur: Time has an orientation, and therefore these paths will be weakly increasing coordinate-wise; a little bit like in relativity theory. Moreover, there are regions in this 3-dimensional state space that are forbidden, due to coordination constraints: Only one, or at most a limited number of processors can access the same piece of memory at the same time.



Directed path in a cube with obstructions. This path is homotopic to a directed path on the boundary of the cube, but not through any directed homotopy

More abstractly, we want to study directed paths in a "space with holes", often modelled as a cubical complex. Combined with a language, these are called Higher Dimensional Automata. It turns out that directed paths that are homotopic to each other, in a directed manner, describe equivalent compound processes, always yielding the same result for a distributed algorithm. A directed homotopy between directed paths, and more generally between directed maps, is a one-parameter deformation where all intermediate steps are directed as well.

Directedness is the essence and the challenge. You cannot apply standard techniques from algebraic topology right away. We found examples of directed paths that are homotopic in the standard sense, but not so when you require directedness for all intermediate paths. In the end, you want to describe and structure the space of all directed maps up to directed homotopy, and to calculate it algorithmically.

CFS: Have people in theoretical computer science found an interest in your approach?

MR: Some have, and one of our first foundational papers still receives citations on a regular basis. But let me admit that there is a larger community in computer science working on coordination problems and algorithms using so-called Petri nets, an area that I do not know a lot about. Many of their and our results resemble each other. In fact, a Dutch collaborator of ours, Rob van Glabbeek, now in Scotland after many years in Australia, has shown in an abstract way that Higher Dimensional Automata are at least as expressive as Petri nets: all that they can model, we can as well.... But we are a small community compared to others.

I would like to add that we consider ourselves as a branch of the rapidly developing area applied topology, with topological data analysis as the most important and popular family member.

### Abel Prize Interviews

CFS: This is all very exciting. Now we are going to switch topics entirely. As you mentioned, you wound up as associate professor and then professor with special responsibilities at Aalborg University. You have been heavily involved with teaching, and you have worked as a supervisor for many students in a wide variety of mathematical topics. You also became Teacher of the year in 2018 and 2019.

Besides all that, you have been involved with public outreach, having given public lectures on several aspects of mathematics. Raising public awareness of mathematics is not a mere slogan for you. You have taken it very seriously. And as one of the editors, and editor-in-chief from 2002 to 2003 of the newsletter Matilde of the Danish Mathematical Society, you published interviews with various mathematicians, which made you eminently prepared to take the initiative for the interviews of the Abel Prize laureates. Could you please talk about how the interviews with the prize laureates came about?


Final issue of Matilde, 2023

MR: Matilde was the journal of the Danish Mathematical Society. It was founded while Bodil Branner was the chairman of the Danish Mathematical Society, which this year is celebrating 150 years of its existence. One of her many initiatives was to establish a regular newsletter. By the way, I came up with its name, Matilde: Mat (matematik in Danish comes without an h) with a tilde!

I became part of the team of newsletter editors, and my task was to interview some of the "grand old men" in Danish mathematics, for example Ebbe Thue Poulsen from Aarhus and Bent Fuglede from Copenhagen. A special treat was an interview with Ib Madsen shortly before his 60th anniversary.

Concerning the Abel Prize, Matilde's editorial board was contacted by the embassy of Norway in Denmark: Would we be interested in covering the first Abel Prize ceremony in Oslo? I discussed the option with Mikael Rørdam, who was the editor-in-chief at the time. I gathered all my courage and suggested that we should ask for an interview with Jean-Pierre Serre, the first recipient of the Abel Prize.

CFS: That was in 2003?

MR: Either in late 2002 or in early 2003. We wrote back to the embassy, and they contacted the Abel committee. The committee was positive and informed professor Serre. Your colleague Kristian Seip must have heard about the initiative...

CFS: He was the chairman of the Norwegian Mathematical Society at the time...

MR: And he thought that it was a pity that the Norwegians had not asked for that opportunity themselves. I do not know whether he asked you first...

CFS: He asked me first, I think.

MR: OK, he suggested that you were involved, as well. I do not remember the details, but the two of us were put in contact. We did not know each other personally at the time. I had participated in the 20th Nordic Congress of Mathematicians in Trondheim in 1988 that you were involved in, but I do not think we made contact then.

Anyway, we teamed up, and that turned out to be a very good idea. First of all, it made me feel more relaxed, and in the long run, we became a "dream team", right? We brought different angles to the interviews, complementing each other. We coordinated the questions we wanted to ask, the last time being in the evening before the interview took place.

CFS: In fact, you taught me a lot about interview techniques. I learnt a lot from you.

MR: Thanks! But you always came up with a lot of interesting questions and citations yourself. I still remember how nervous we were when we started the first interview with Jean-Pierre Serre.

CFS: And he was not in a good mood initially!

MR: He was not, and I kind of understand why: The Abel Prize recipients have all been quite old so far. During the week of festivities, they have a strenuous program. They are asked to attend a lot of meetings, ceremonies, they must give talks, and then one or several interviews! I do not know, but I imagine that Serre thought: "Another interview with silly journalists asking me silly questions."

CFS: That's right. But let me tell you, the real breakthrough came when you told him that you were aware how he discovered some very important notion having something to do with fibre bundles. Then he lit up immediately. "Oh, you knew about that?", he said, and that changed the atmosphere. He became very positive from then on.

MR: Mentioning the path fibration changed the game. I re-read the interview this morning, and it is, in fact, still very interesting!

CFS: We should not underestimate the importance of the initiative of yours getting started with the Abel interviews. The two of us went on and conducted the Abel interviews for 14 consecutive years. Then I continued together with Bjørn Dundas. I would say unabashedly that the Abel interviews are very important and successful. They are published both in the European Mathematical Society Newsletter, now Magazine, and in the AMS Notices. Besides, the European Mathematical Society's publishing house, EMS Press, published a book with the interviews the two of us had.

MR: I am also happy having been a part of that enterprise. In the end, looking back at my career, that may have been the most important thing I have been dealing with!

Abel interview 2016: Sir Andew J. Wiles, Martin Raussen, Christian F. Skau. (Photo: Eirik F. Baardsen, DNVA.)

CFS: To make the Abel Prize known for the maths community and beyond, these interviews have been very important, I think it is fair to say that!

Could I ask you, and this is a very difficult question: Do you have some special highlights that you would mention from all these interviews?

MR: Let me try. I have two different answers to your question, on very different levels, though. The first answer is related to the amount of the Abel Prize.

CFS: By the way, the prize money was almost one million US dollars.

MR: Right. Often, we did not dare to ask what the recipients would do with the prize money, because that is quite personal. But I remember that we listened in on the interview conducted by a journalist with professor Serre. This journalist was less shy and asked Serre directly what he would use the prize money for. His answer was just laconic: “Well, I'll see to have it spent.”

Another occasion that I remember vividly, occurred when we interviewed Pierre Deligne, ten years after Serre. He had made it already public, and he also told us at the beginning of the interview, that this was not money for him, but it was money for mathematics. He wanted to give it to various institutions that had played an important role in his own career: the research institute IHÉS in Bures-sur-Yvette in the suburbs of Paris, and the IAS in Princeton, USA. But also to two institutions in Russia, the Department of Mathematics of the Higher School of Economics, and to the Russian Dynasty Foundation, which supported science. Deligne was clearly one of the most modest interview subjects we had.

My second answer to your question is related to reactions of our interviewees to one of our standard questions: Did some of their great results rely on sudden flashes of insight, at least partially? We have all heard the story, published in a book by Hadamard, of Henri Poincaré who while stepping into a bus somewhere in Northern France, in a sudden flash of insight saw connections between modular forms, Fuchsian functions and so on, and formalisms in non-Euclidean geometry on the other side. This has become an iconic story, an incident where a new connection suddenly pops up after the sub-conscience has worked on it for some time. Stories of that kind have fascinated me.

Getting back to the interview with Serre: Around 1950 he worked on homotopy groups of spheres, and he established many of their properties. One of the techniques applied in the investigations makes use of the path fibration. It relates the homotopy groups of a space with the homotopy groups of its loop space. You can explain it – if you know about it – to a student who knows some elementary homotopy theory in a few minutes. But that relationship had not yet been established, before Serre.

Other far more subtle techniques, for example, spectral sequences (mainly developed by Leray for other purposes), were known and applied by Serre. Moreover, he made calculations one prime at a time, so to speak. Anyway, the path fibration was apparently a missing link that occurred to him on the night train when returning from vacations. He got so excited that he woke up his wife to tell her!

John Tate told us that when he had worked hard on the determination of higher-dimensional cohomology groups in class field theory, he went to a party and had a few drinks, came home and suddenly saw the solution after midnight. I got jealous when I heard that!

But everybody we asked told us that this happened very rarely, perhaps twice or three times in a lifetime. Moreover, they told us that such nice experiences never come for free!

They may occur after a long time and many efforts trying to put things together. Only when you have made these efforts and perhaps feel that you run against the wall, emotions come into the picture, and then your sub-consciousness might do part of the work for you. I remember very well one word that I will never forget: When discussing that same question with the late Abel Prize laureate Nirenberg, he coined it succinctly with a word in my native German: It needs “Sitzfleisch” (seat flesh) – expressing persistence in a very physical manner – to get to anything serious. That is a story that I also like to tell my students!

CFS: This reminds me of the following story about Niels Henrik Abel: He visited Berlin and stayed with some Norwegian friends at a shared flat. One of these friends wrote later that Abel would often wake up in the middle of the night, light a candle, and scribble down some mathematical ideas that had occurred to him during his sleep. I would certainly have appreciated it if that had ever happened to me!

MR: So would I!

**Activities in the European Mathematical Society**

CFS: At the time the first Abel Prize interview was taken you were headhunted to the Newsletter of the European Mathematical Society (EMS), where you were editor-in-chief from 2003 to 2008. And in 2009, you were voted into the EMS executive committee (EC), where you participated from 2009 to 2016, the last six years as Vice President. You were liaison for the EMS committee Raising Public Awareness, and you were also the person in charge of the EMS web profile. All these are heavy and time-consuming duties. You were still teaching, and you didn't get much credit as pertains to your teaching load. Could you comment on this period of your life?

MR: It was a challenging period that did consume quite a lot of my time, which was "stolen" from my research and from my family, I must admit. But on the other hand, I liked these duties, as well. I could utilize other facets of my personality, of my capabilities. And when I left the executive board of the EMS, I missed it for a while. It had almost become a second family for me. Most of my colleagues at the EMS are highly devoted. When they take on a job, they really do so reliably. I felt that we were pulling on the same string.

You see, the EMS is still a relatively new society. It was founded in 1990, a newcomer in comparison with the Norwegian or the Danish societies. It has almost all European mathematical societies as members, but there are still all too few individual members, only a bit more than 3000. Since I started, membership has risen from 2000 to 3000, but it's still not living up to its potential within all of Europe!

At the board meetings, discussions and decisions have to be taken all the time. For example, how is the money spent? Not that the EMS has a lot of money, but it can give something to organizations or to conferences, workshops and so on. How are you going to find out who is worth giving the money to? I was a member of the society's meetings committee preparing some of these decisions. By the way, the EMS executive committee nominates some of the members of the Abel Committee that selects the Abel Prize winners.

But probably it is more important to develop contacts with some of the politicians, speaking about the importance of mathematics towards Brussels, towards the people who really can give substantial support. That's a tough challenge. I've not really been much involved in this task. That was the challenge of the presidents of the EMS. They have done tremendous work in that direction, sometimes with success. But of course, you could always hope for much more.

I think it's really important that we have such a transnational mathematical society. Local issues can be handled much better by a national mathematical society. But it's also important to have a player on the transnational scene, to promote and to get inspiration from each other, and also to have a counterweight to the American Mathematical Society. I do not want to be negative about the American Mathematical Society at all, but it is such a dominating society, mainly for good. On the other hand, if you didn't have a competitor with the same goals, they would take over everything. A concrete instance is the database MathSciNet, a highly valuable tool. It is very important that we also have Zentralblatt/zbMATH, because otherwise there would only be one venue, and you know, when there is no competition... The EMS is one of the owners of Zentralblatt, and there is now free access!

CFS: How many members are there in the American Mathematical Society?

MR: I would say around 30,000. And of course, I'm also a member of the AMS, and most European mathematicians are, but too few are members of the EMS, I would say.

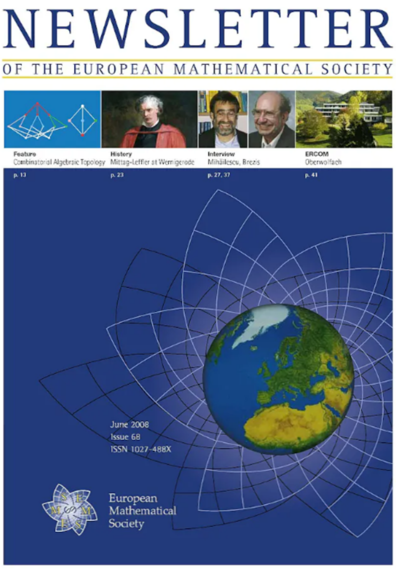
CFS: I observed that there were disappointingly few members at my department in Trondheim.

MR: Perhaps some of the readers of this interview get the idea. It's in fact cheap to become a member!

**Raising Awareness of Mathematics**

CFS: I already mentioned that you have been a spokesperson for raising awareness of mathematics. Why is that an important endeavour?

MR: Well, for several years, from 2003 to 2008, I was the editor-in-chief of the Newsletter of the EMS. That newsletter was mainly read by mathematicians. I think that also our colleagues need good stories about successes in their own discipline!




Newsletter of the EMS, 68/2008

When it comes to communicating the importance of mathematics to laymen and non-mathematicians, a lot of good work has been done by the Raising Public Awareness Committee (RPA) of the EMS. Over the years, RPA was headed by Vagn Lundsgaard Hansen from Denmark, Ehrhard Behrends from Germany, and Roberto Natalini from Italy. Members of that committee write popular books, coordinate web events, and perform "Mathematics in the Street" events on several occasions. While I was on the executive committee (EC) of the EMS, I had the role of liaison officer linking the EC with the RPA Committee, and I participated in several of its meetings. I am impressed with the work they did, and do, but I have not been very active in that direction myself.

Generally speaking, promoting mathematics seems to be more demanding than doing it for many other disciplines. One reason is arguably that mathematics is a subject that everybody has been in touch with at school, from grade one on. Hence, many people think they somehow know what mathematics is about. Depending on the level of their education, they might connect mathematics with calculations involving large numbers or evaluation of difficult integrals.

Quite few people are aware of the fact that mathematics is much more far-reaching, consisting of many subdisciplines and having so many facets. It has its own touch of beauty, it has philosophical components, and, on the other hand, it has become increasingly important in technological developments underpinning everyday life.

In comparison, physicists or computer scientists have an advantage: Their subjects are taught to a much lesser extent in school, and people realize that it is worthwhile to know more. Moreover, it is easier for them to come up with compelling pictures, for example those from the Hubble or the James Webb telescopes showing us the universe in its splendour. That does not mean that a layman like me fully understands modern physical theories, but these pictures fill you with awe. It is not that mathematicians do not try to do something similar: There are lots of beautiful and impressive pictures available on various platforms, like for example IMAGINARY2. But it is more difficult to demonstrate to the uninitiated that coding theory or topological data analysis are built on beautiful insights with tremendous applications.

CFS: I think this is a very important point.

MR: So, let us welcome every initiative to build bridges! My own contribution has been directed to the mathematical community. The Abel Prize interview conveyed essential points of view of extraordinary mathematicians.

As a teacher, I try not only to talk about interesting and important mathematical concepts, results, and techniques, but also to put things into perspective. What is the reason for studying a particular topic? Why is this result a small technical lemma, and why is another one an important theorem? For example, why is Gauss' Theorema Egregium really "egregium"?

CFS: It is really depressing that many people are almost proud of admitting that they are illiterate in mathematics, or that they even hate mathematics! Take for example Aftenposten, the largest Norwegian newspaper, which is supposed to have a cultural dimension as well. Nevertheless, they hardly mention the Abel Prize at all. In the twenty years of the existence of the prize, very few articles have appeared about the Abel Prize and its recipients. Without any shame, they have neglected it over many years!

MR: We have also had great difficulties to advertise our Nordic Congress in the press. The local newspaper that often boasts of minor local initiatives neglects the congress. The only laudable exception is the newsletter Ingeniøren, published by the trade union of Danish engineers.

## Beauty and Importance of Mathematics

CFS: In this connection, I would like to read an excerpt from the eulogy that the brilliant Norwegian mathematician Ludvig Sylow wrote in 1899 on the occasion of Sophus Lie's death: "It is the mathematician's misfortune more than the other scientists that his work cannot be presented or interpreted even for the educated general public. In fact, hardly to a collection of scientists from other fields. One has to be a mathematician to appreciate the beauty of a proof of a major theorem, or to admire the edifice erected by mathematicians over thousands of years." Any comments?

MR: Well, to assess that something is exceptionally beautiful, you need to have certain background knowledge. But this is also true for advanced music or painting. I think, though, that it is easier for artists to get us emotionally involved. It is far more difficult to achieve this when talking about mathematical ideas and methods. People have tried to find different ways to do that. It is a challenge for the entire community.

CFS: That reminds me of a statement by one of the Abel Prize recipients, the late John Tate, who stated during our interview: "Mathematics is both art and science. There are artistic aspects to mathematics. It is just beautiful. Unfortunately, it is only beautiful to the initiated, to the people who do it. It can't really be understood or appreciated much on a popular level the way music can. You do not have to be a composer to enjoy music, but in mathematics you have to be a mathematician to appreciate it." Another Abel Prize recipient, László Lovász, lamented that science in general is in danger. Things have become so complicated that it is very difficult to distinguish between science and pseudoscience. Mathematicians, and scientists at large, have a responsibility for making the public aware of this danger, which has wide implications.

MR: That is already a challenge at school!

CFS: Lovász thinks that we do not teach mathematics the right way in high school, at least seen from his Hungarian perspective. We do not give pupils enough safeguards against pseudoscience and unscientific speculation.

MR: Hard to disagree! But not everything is bad: For centuries, mathematics had little influence on technology, on real life. Of course, there are examples to the contrary, land surveying in Egypt and ancient astronomy, for example. But the applicability of mathematics has exploded, especially in our lifetime. Mathematics is no longer only essential for physics and engineering, mathematics is everywhere! It has become an economic pillar of society, underpinning important industry sectors, going hand in hand with engineering and computer science. Many consumer goods, modern cars, planes are inconceivable without the input of mathematics. Just think about our means of communication, and credit cards, GPS, as well. Mathematical modelling makes reliable predictions possible. No data science without mathematics!

Number theory was completely estranged from applications, as Hardy proudly noted less than a century ago. Now sophisticated number theory, deep results on elliptic curves, are quintessential in coding and cryptography. But unfortunately, the mathematical community has not been good enough to communicate the beauty and the practical importance of our subject.

CFS: Tell us about the quote from René Thom that you mentioned prior to this interview!

MR: I told you earlier that I spent a year in Paris as a Ph.D. student. I went regularly to the IHÉS at Bures-sur-Yvette. At that time René Thom was still around, probably as an emeritus professor. He was a demigod in my eyes; he had developed so many important parts of differential and algebraic topology. At that time, his catastrophe theory was still a hot topic. Thom had by then acquired a deep interest in the history of mathematics. On Saturdays, when the institute otherwise was quite calm, he ran a seminar on the history of mathematics, with varying lecturers. I went there several times. During one of the sessions, he said (slightly paraphrased): "I was told that more than half of the mathematicians throughout history are still alive. But that is certainly not the better half!"

CFS: I like that quote! Now a quote from Harish-Chandra: "I have often pondered over the roles of knowledge or experience on the one hand, and imagination or intuition on the other, in the process of discovery. I believe that there is a certain fundamental conflict between the two. And knowledge, by advocating caution, tends to inhibit the flight of imagination. Therefore, a certain naïveté, unburdened by conventional wisdom, can sometimes be a positive asset." Do you have comments?

MR: It is certainly true that the most stunning results of the Abel Prize recipients that we interviewed go back to their young age, when they were around thirty years old, or younger. In that sense Harish-Chandra seems to be right. Our interviewees told us that when they got older, they knew the literature better and also how to avoid possible pitfalls. But some naïveté is needed, as well as brute force, to start on a mathematical endeavour, thinking: "Perhaps I can do something here where nobody else so far has been successful."

At a higher age, these top mathematicians are of course still enormously talented. They would build on previous experiences, on their contact network, and advance the subject by sharing ideas with others. But the naïve courage to start on a new undeveloped area with brute force is rather something for talented young less experienced people.

CFS: This brings us again to Hardy's "Mathematics is a young man's game", that you just explained very eloquently. And still, you need "Sitzfleisch"!

## Computer Aided Mathematics

Now to a hard question: If you should venture a guess, which mathematical areas do you think are going to witness the most important developments in the coming years?

MR: That sounds like a question for people from the Champions League, and not to me.

CFS: Let me give you a clue: Could it be that using computers will have some effect?

MR: Well, they have already had a certain effect. I mean, it started with the solution of the four-colour problem back in the late seventies. And it happens more and more often that results are checked with computer aid. And there are new developments. Already several years ago the proof of the Feit–Thompson theorem about the solvability of groups of odd order was checked by computer, and that proof fills a 250 pages journal paper. I've heard by hearsay that the proof of an important step in the new discipline of condensed mathematics advanced by Dustin Clausen and Peter Scholze needed to be checked by computer; this was done successfully in an effort that applied the proof assistant system Lean.

I imagine that future mathematicians will get a lot of help from advanced artificial intelligence. Already now you can ask whether there exist results in a certain direction. And then the entire existing literature will be searched. If you ask intelligently enough, then you will either find an answer, or a "no". I guess that you can progressively enter a dialogue with computers, adding two forces: human imagination on the one hand, and the enormous power of computer algorithms searching through and combining everything that exists, and to do that very quickly, on the other hand. And I don't see an end to that story.

When I was young, it was completely unimaginable to think that a computer might be better at chess or Go than a human being. But now they are!

CFS: It's sort of depressing in a way.

MR: On the other hand, maybe it opens venues and options that you otherwise couldn't be aware of.

CFS: That is certainly true. When we interviewed Peter Lax, who is also one of the Abel Prize recipients, about how fast computers had become, he said that half the speed is due to clever algorithms. But to get clever algorithms you need mathematicians.

MR: And the work developing these algorithms for computer aided mathematics demands mathematicians at a high level.

## Frogs and Birds

CFS: Now to a quotation from Felix Klein. He said that mathematics develops as old results are being understood and illuminated by new methods and insights. Proportionally with a better and deeper understanding, new problems naturally arise. Your comments?

MR: I remember we asked some of the Abel Prize recipients about that. One might be afraid that at a certain point everything that is worth developing in mathematics has already been done. How could a young person then find something worthwhile to think about? But I think it was Serre who said something like: "I don't worry about that. New problems do arise all the time." Look at what happened during our lifetimes: So many new subjects in mathematics that almost didn't exist when we started, are flourishing now. Others, that were very much the talk of the day do still exist, but not with the same attention. I think it's particularly young people who can somehow sense in which directions new things happen, and then they dive into these and give them a push.

CFS: Hilbert said that problems are the lifeblood of mathematics. No question about that.

MR: I remember that we received different answers to the question, whether our interviewees regarded themselves mainly as theory builders or as problem solvers. You have these two capacities within one person, but more widely, also in the mathematical community. It is very important that both types of mathematical preferences coexist, and that they can profit from each other.

CFS: You gave me the clue to another quotation, this time from Freeman Dyson. I think he describes "theorem builder" versus "problem solver" very well: "Some mathematicians are birds, others are frogs. Birds fly high in the air and survey broad vistas of mathematics, out to the horizon. They delight in concepts that unify our thinking and bring together diverse problems from different parts of the landscape. Frogs lie in the mud below and see only the flowers that grow nearby. They delight in the details of a particular object, and they solve problems one at a time." I think that's a very nice description.

MR: I think it's almost impossible to say that in a better way. It's important that the frogs and the birds coexist.

CFS: I agree with you. I think it was Andrew Wiles who told us in the interview: "The definition of a good mathematical problem is the mathematics it generates, rather than the problem itself."

MR: That's well said. And there are many instances where you can show that this is true. Take the Fermat conjecture as an example. Due to Wiles, we know that it is true. But the whole build-up leading to the final solution is so much more important than the result itself. It has given insights that Fermat could not even have dreamt of.

CFS: Perhaps it is relevant here to mention, as we did in the interview with Atiyah and Singer, Eugene Wigner's statement about the unreasonable effectiveness of mathematics in the physical sciences, as well as Galileo's dictum, that the laws of nature are written in the language of mathematics.

**Private Interests**

Let us end this interview with a question that you also suggested including in the Abel interviews: What interests do you have outside of mathematics?

MR: My answer to that question might be a little disappointing. Of course, I must think about that now because I'm going to retire, and that means that mathematics will play a minor role in my life in the future. Hobbies will have to take over more time. I would have difficulties to single out a specific hobby and say: "Now I know exactly what I will spend my time on." In a more positive way, and certainly exaggerating, I would say I'm a renaissance man in the sense that I have many things that I'm interested in.

I like to read popular science books, and I hope to get more time to do that. Biology, genetics, evolution, climate, so many new things happen. I have often looked avidly into the leaflets from an institution called People's University and told myself: "Okay, you could do this, and you could do that if you just had the time." Now I will get more time to spend on these interests.

Moreover, I like listening to and learning about music, both classical music and jazz. If it were not for this congress, I would have gone to many events at the Copenhagen Jazz Week, which is going on in this same week. I'm now living most of the time in the Copenhagen area, and I take the opportunity to go to classical concerts by the Radio Symphony Orchestra and the like from time to time. I also like to visit art exhibitions. Perhaps I might play a bit more the piano as I did as a teenager, but not systematically ever since. Last but not least, more time for my family!

CFS: Thank you very much on behalf of myself and of the Norwegian and the Danish Mathematical Society for this very interesting interview. It's been a pleasure.

MR: It has been my pleasure as well. And I'm thrilled and thankful that you took the effort to come to Aalborg just to interview an ordinary mathematician like me.

# A Research Career in Focus

*Horia Cornean*

Reflecting on my work in mathematical physics, I see that my path has been largely shaped by an interest in the underpinnings of quantum mechanics and solid-state physics, particularly through the lens of spectral theory. My research has always been about pushing the boundaries of what we understand about the quantum world, focusing on the practical application of complex mathematical concepts to physical phenomena.

One of the main areas I have concentrated on is the study of Schrödinger operators, which are crucial for understanding the energy levels within quantum systems. These operators form the backbone of quantum mechanics, and my work has aimed at providing a clearer picture of their spectral properties. This focus has been particularly relevant for exploring the behavior of electrons in solid-state physics, especially when considering the effects of magnetic fields and disorder.

Another significant part of my research has involved topological insulators. These materials have an intriguing property of being conductive on the surface while insulating in the interior. My interest here lies in the mathematical exploration of their band structure's topological aspects. Understanding these materials' topological invariants has been key to unraveling their unique conducting properties, which could have implications for quantum computing and electronics.

I have also delved into the mathematical models behind the quantum Hall effect, a phenomenon seen in two-dimensional electron systems under certain conditions. This area of study has challenged me to explore the quantum Hall effect's mathematical foundations, particularly the role of edge states and conductance quantization.

Collaboration has been a vital component of my research approach. The nature of mathematical physics is inherently interdisciplinary, requiring a blend of insights from different fields. Through collaborations, I have sought to engage in a meaningful exchange of ideas, contributing to the broader scientific dialogue. Additionally, mentoring has been an integral part of my career, and I believe in the importance of guiding young researchers, sharing knowledge, and fostering a supportive academic community.

Looking back, my journey through mathematical physics has been about seeking a deeper understanding of the quantum realm, bridging theory with practice, and contributing to our collective knowledge. It is a journey marked by curiosity, collaboration, and a commitment to education and mentorship.

# External Funding: Curse or Blessing?

*Rasmus Waagepetersen*

When I started my career at Aalborg University in 2000, my work assignments were mainly research and teaching. At this time, applying for external funding was not considered a task for junior staff and mainly 'icing on the cake', which was nice, but not indispensable. This has since changed dramatically as reductions in basic funding, at least relative to the number of researchers, imply that the universities are relying much more on external funding to make ends meet today. Therefore, it is expected today that academic staff engage actively in the acquisition of external funding, and it is probably fair to say that this is not a welcome change among the majority of researchers. One main reason is that acceptance rates for applications are generally very low, often less than 10%, meaning that writing applications may seem quite futile.

Looking back on the past five years, I have been engaged in 16 applications, either as a main applicant or co-applicant, and four of these were successful, resulting in a success rate of 25% of the total number of applications. However, the success rate regarding actual acquired money is considerably lower, since some applications for large amounts of money were rejected. Moreover, the success rate was zero for several years, which was quite discouraging. The applications have concerned various research topics, ranging from basic research in spatial point processes, my main field of expertise, to interdisciplinary research in the decay of batteries, and the proficiency of arithmetic concerning fractions in primary schools.

Curiously and paradoxically, I have yet to acquire money for research in the field where I am a true expert. The funding I obtained was mainly based on being a seasoned statistical researcher with general statistical expertise; whether this should be considered good or bad depends on personal preferences. On the one hand, it is sad not to be able to sustain a research stronghold resulting from many years of hard work, but on the other hand, I find it refreshing to engage in new and interesting research problems. However, one thing is certain, the freedom of research for each individual academic is under pressure: Your research topics are very much controlled by the seemingly more or less random outcome of applications. Furthermore, by working on too many very diverse projects, there is a danger of the researcher not obtaining deep research expertise within a given field, which may prevent profound breakthroughs in the future.

So, what am I working on these days that is being funded externally? One project concerns developing and testing a new teaching methodology in mathematics, inspired by teaching approaches in mathematics in Singaporean primary schools. A large randomized experiment, at 120 schools, keeps track of test results in mathematics for pupils during their $4^{\text{th}}$ through $6^{\text{th}}$ school year: Approximately half used the current approach and the remaining half used the new methodology. My role is to analyze the data using mixed models to account for the hierarchical structure of the data, i.e., tests between students within class within school. Furthermore, latent growth models may be used to study the development of test results over time. Another project is the Ph.D. project of Emilie Højbjerre-Frandsen, which also concerns randomized trials but with applications in medical science. Here, the objective is to increase the power of clinical trials by utilizing historical data to build strong covariates that can reduce the noise levels in current clinical trials, which must be done without compromising type-I error control.

The idea of the Villum Interdisciplinary Synergy project UrbanLab is to explore how modern spatial statistics methodology can be used to exploit high-resolution geographical data to investigate socioeconomic hypotheses related to housing policies. While econometrics is an advanced branch of statistics, it is clear that most econometricians are unfamiliar with spatial statistical methodology. The aim of the study is, therefore, to provide a framework upon which the fields of sociology, economics, and spatial statistics can benefit from each other.

Speaking about benefiting from each other, the last project that I am involved in, NSECURE funded by the Novo Nordisk Foundation, has the ultimate goal of enhancing the root microbiome of wheat plants to make the plants self-sufficient with nitrogen. This could diminish the use of artificial fertilizers which is detrimental to the environment and a considerable economic challenge in developing countries. Essentially, the root microbiome is the complex community of microorganisms that reside inside or around the roots of plants. Some bacteria can harvest nitrogen from the atmosphere and make the nitrogen available to the plant. This project hopes to design plant microbiomes that contain these bacteria and, importantly, where these bacteria can flourish without being outcompeted by other bacteria. This requires a deep understanding of the dynamics of the microbiome and the causal relations between different types of bacteria. From the statistical point of view, this calls for the development of a time series methodology for highly multivariate compositional data and the design of experiments that can reveal causal relations.

All the projects mentioned above are very interesting and address important societal challenges. It is also clear that the projects are very different, which can be mentally challenging. Overall, I find it rewarding to engage with these projects and the derived potential for getting a broader perspective of statistical science. However, I have, as a young researcher, already had the privilege to deeply engage in a research topic of my own choice without having to worry about funding, which is unfortunately not the case for young researchers today!

# Mathematics at University

*Jacob Engberg*

The following article is a foray into what mathematics, among friends, at university looks like. As a disclaimer, I can only speak about how I was taught mathematics, but I imagine that much of what is covered here is similar to how mathematics is taught in the rest of Europe and maybe even the U.S. I am unfamiliar with how the subject is taught in the rest of the world, but I expect certain elements to carry over. I hope these pages still warrant your time and attention. At the time of writing, I am studying mathematics at the university level, and every opinion expressed in the following article is mine alone. Most of the mathematics (hereafter abbreviated math) in these pages comes from *Funktioner af en og flere variable* by Ebbe Thue Poulsen and *The Triangle Inequality* at *Proofwiki.org*.

**Introduction**

I imagine that you are reading this article because you are interested in how math is done at the university level. In my social circles, at the university and online, I hear a sentiment repeated a lot: When people are taught math at the university level, they are initially surprised because it is quite different from what they expected. Sometimes, what they expect to find is closer to how math is applied in subjects within engineering. In my other social circles, outside of my education, I also notice an uncertainty in familiarity with the specifics of an education in math. I will, therefore, touch upon what I do day-to-day, related to my studies.

I am also hoping that you might be left with a certain 'feeling': The feeling of what it is like to 'do math'. There is a 'grittiness' to problems in math: It is strenuous to read, write, and understand, and you are fighting with it until a light bulb just goes off. Then you suddenly 'get it'; because of course! It is supposed to be done this way, or these things are true because of that other thing, etc.

The math discussed in these pages will be math taught to any student participating in a version of a Real Analysis course. I am going to try to make everything understandable for someone who has not touched math since 9$^{\text{th}}$ grade, but I will not be prevaricating around either - all math shown here will be something I was taught, in a real lecture room, using real notation, studying math at the university. It will be 'Real Math'.

**The Term 'Real Math'**

Amongst mathematicians, and sometimes online, you might hear the term 'Real Math', which is supposedly the thing taught at a university math degree. The mathematics that most people are probably going to be familiar with concerns calculating or arithmetic: You set up some sort of a problem with a bunch of numbers, and then you crunch those numbers, either in your mind or using a computer, and you obtain some sort of result. As this gets more advanced, we move into the field of engineering: Say you want to build a bridge. You want it to have a certain length and use certain materials and techniques. You also want to know how much of the materials you will need, what load the bridge can take, and the various forces acting on the bridge and its surroundings. These are all questions about design and calculation. When you then do the number crunching and arrive at a final number for each of your questions, the field as this pertains to is usually engineering. This is not to say that 'Real Math' does not have a lot of number crunching, just like when people think of the math they know and try to imagine an 'advanced' version of that, they will probably end up thinking of engineering.

When people talk about 'Real Math', they are usually talking about theorems and proofs. In 'Real Math', the goal is not to do number crunching to arrive at a specific result in the form of another number, but rather to prove something, unequivocally, through the use of logical steps. In the purest form, it goes something like this: We start with a set of things we assume are true in the mathematical world: These are called axioms: An axiom could be that for any decimal number, you can find a larger integer. Next, we formulate a theorem; A theorem consists of a set of assumptions and some conclusions that may follow. If we assume A, B, and C, a result D is given. To prove that D follows, we start with our assumptions, and through the use of correct and logical steps, we arrive at the conclusion. There are many types of proof, depending on what needs to be proven, which can be researched by those who wish.

The real analysis course I mentioned earlier is the place most people will encounter 'Real Math' for the first time, and this article will therefore introduce 'Real Math' within this context. Moreover, there is a lot more, and much more abstract, math out there that is taught at higher levels. Learning, writing, and understanding this process of definition(s)-theorem-proof and variations thereof, repeated ad nauseam, is essentially what 'Real Math' is all about. To clarify, I usually consider a lot of examples of the application of theorems in practice, solve exercises using the theorem, or prove related theorems. The specifics usually depend on the course, lecture, or project. In the next section, I will exemplify this and show some of the 'Real Math' that I was taught.

**The Math**

To particularize the aforementioned 'Real Math', I will introduce what is meant by a 'function' and a concept related thereto. Next, I will introduce a theorem and prove it. Note that I will be using some mathematical notation, which you would be expected to know at the university level, and explain these as I go along in varying degrees. To begin, pre-university students of math may be familiar with $<$ and $\leq$, called 'less than' and 'less than or equal'. These act exactly according to their description and inform the student whether a number to the left is 'less than' or 'less than or equal' to the number on the right. To exemplify the use of these symbols, the following statements are all true:

$$1 < 2,$$
$$1 \leq 2,$$
$$2 \leq 2.$$

whereas the following statement is NOT true:

$$2 < 2.$$

Both symbols have an equivalent, namely $>$ and $\geq$, which are called 'greater than' and 'greater than or equal', and behave as you would expect.

The notation $|x|$ is called 'the absolute value', which consequently suggests that its value is always positive, e.g. $|-2| = |2| = 2$. In the context of this article, it is used to define a distance: See the figure below and look at the number 1:

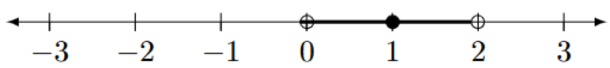



**Figure 1:** A number line

The number 2 is one step away from the number 1: There is a distance of 1. No matter whether you start from the number 1 or from the number 2, you have to take a single step to get to the other number and so:

$$|1 - 2| = |-1| = 1,$$
$$|2 - 1| = |1| = 1.$$

We could also ask: If we stand at the number 1, what numbers are within a distance of 1? To answer this question, we look at some number $x$ and use the notation of the absolute value introduced earlier. If $x$ satisfies the inequality

$$|x - 1| \leq 1, \tag{1}$$

then the number $x$ would be within a distance of 1 to the number 1. Once again, we can reaffirm our intuition by looking at the figure above, where the number 0 could be $x$; but, all the numbers between 0 and 2 ($0 \leq x \leq 2$) could also be $x$, and we simply use the inequality above to make sure that a given $x$ is within a certain distance of the number 1. In the same sense, we can describe the notion of distance as a tolerance: Any $x$ is accepted, as long as it is close enough to the number 1. In this case, 'close enough' would be a distance of at most one. The table below contains a list of symbols of notation that we will consider next:

| **Symbol** | **Meaning** |
|---|---|
| $\forall$ | For All |
| $\exists$ | There Exists |
| $\in$ | In |
| $\implies$ | Implies |

The symbols above are used per their attributed meaning, i.e. you can replace the symbol with their respective phrase of meaning, e.g. $\forall$ means: "For all *something*..." and $\exists$ means: "There exists *a thing*...". Usually, these are used in combination with each other, e.g:

| | | | | | | |
|---|---|---|---|---|---|---|
| We write: | $\forall$ | things | $\in$ | something, | $\exists$ | another thing. |
| Which means: | For all | things | in | something, | there exists | another thing. |

The reason these symbols, and many like them, have been defined is a need for precision and rigorousness as best described by a quote from one of my lectures: "You can't talk imprecisely about something precise". This precision allows us to be rigorous: when every symbol has a precise meaning (and you know that meaning), you can follow a proof step by step, and arrive at the same conclusion as the author, or you are able to point out exactly where the proof fails. This is why mathematicians are obsessed with rigor. Rigor means that we only follow and use previously proven things, and every logical step we take or conclusion we draw is therefore also correct. Using and understanding rigor properly essentially takes a degree in math. Still, it would be dishonest not to attempt to be rigorous, when the point of this article is to show what I do on a daily basis. This is also the reason why I have introduced the symbols earlier so carefully, as you would, by studying a degree in Math at the university level, be expected to remember these by heart.

**Functions**

Rigorously defining a function and its properties could almost be considered an entire field in and of itself, and thus, we will restrict ourselves to discussing only its most necessary aspects. The definition of a function relies on 'sets': A 'set' is just a collection of mathematical objects, often numbers, and by extension, the following are all sets:

$$\{1, 2, 3\}, \qquad \{[0, 1], 1000000\}, \qquad \{\heartsuit, 39, \ast\},$$

where the symbol $[0, 1]$ means the interval from 0 to 1 that includes both 0 and 1. The number line is also a set, which we call the 'real numbers': All the numbers you learn about up to and including $9^{\text{th}}$ grade are real numbers. As shown above, sets do not necessarily consist of just numbers but will for our purposes in this article.

> **Definition 2.1.** *Function*
> A function is an entity that takes every element from one set, e.g. $X$, and assigns it to exactly one element from another set, e.g. $Y$. If that function is called $f$, we use the notation
>
> $$f : X \to Y.$$

In many ways, you can think of a function as a recipe, which begins with the ingredient of a number, does some operation to that number, and outputs another number. You might be familiar with the notation $f(x)$, called $f$ of $x$, which we can describe more precisely as follows:

- The name of the function is $f$: This is what we use when discussing the function,
- the number $x$ is the input: The ingredient of the recipe,
- and $f(x)$ is a number: The output of the recipe.

It is common to talk about the 'graph' of a function, which, in our case, refers to this: In a 2D-coordinate system, for every input $x$, a point is drawn with the coordinates of $(x, f(x))$. If you do that for every input $x$, the resulting graph for a reasonable function will be a curve (see Figure 2 below for an example).

**Continuity**

> **Definition 2.2.** *Continuity*
> Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and let $x_0$ be a number in $\mathbb{R}$. If
>
> $$\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall x \in \mathbb{R} \; : \; |x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon,$$
>
> then $f$ is continuous in $x_0$.

We are now going to talk about how continuity is defined. In 7th to 9th grade, you might have heard the term 'continuous', and having it explained as "being able to draw the graph of a function without lifting the pencil from the paper". In our case, that is a fine intuitive understanding, but how would you go about rigorously proving that something is continuous? Moreover, what if we were talking about functions with multiple inputs? Being able to examine these questions is the reason why mathematicians have created these definitions. The following table tries to explain the definition in words.

| **Mathematical notation** | **In words** |
| --- | --- |
| $\forall \varepsilon > 0 \, \exists \delta > 0$ | For all $\varepsilon$ greater than 0, there exists a $\delta$ greater than 0 such that |
| $\forall x \in \mathbb{R}$ : | for all numbers on the number line, the following is true: |
| $\|x - x_0\| < \delta \implies$ | If $x$ is closer to $x_0$ than $\delta$, then |
| $\|f(x) - f(x_0)\| < \varepsilon$ | the function's output of those two numbers will be closer than $\varepsilon$. |

**Remark:** The distinction of "choosing an $\varepsilon$" might not make sense initially, but the reason for using this terminology can be deduced per Definition 2.2, which specifies that a $\delta$ must be found for ALL $\varepsilon > 0$. In the case that we can find an $\varepsilon$ such that we could not find a $\delta$ that satisfies Definition 2.2, the function would not be continuous. It would be a 'disproof by counterexample'. If we instead wanted to show that some function is continuous, we would have to show that for EVERY $\varepsilon > 0$, we could find a corresponding $\delta$. We will do so later.

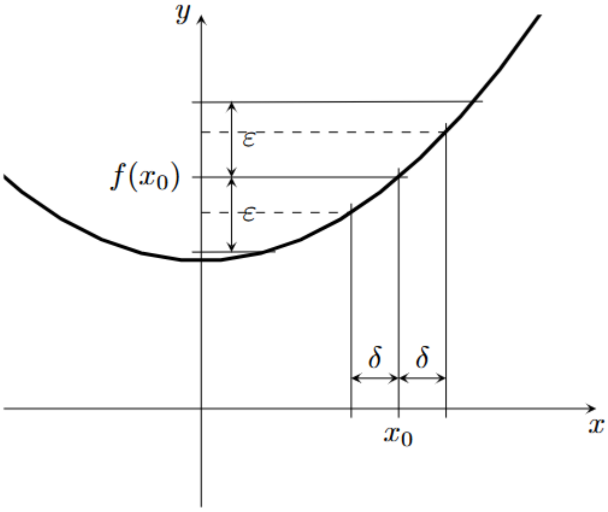
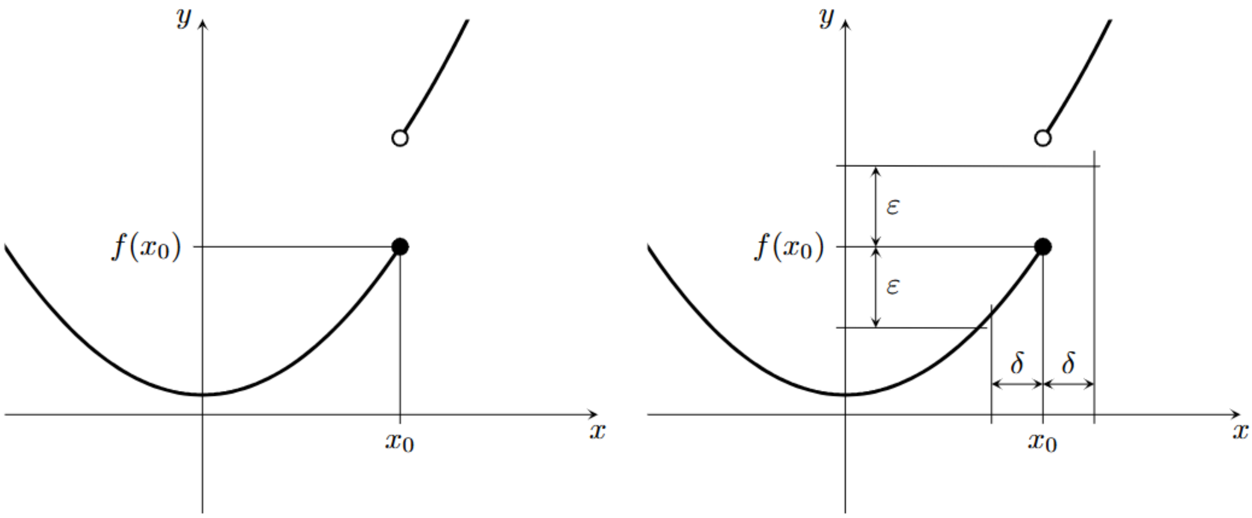



**Figure 2:** A nice, smooth function

Figure 3$a$ illustrates a case in which Definition 2.2 is not satisfied. Obviously, we cannot draw this function without lifting the pencil; anyone can see that! But how does our mathematical definition play into this? In the figure, we can see that there, in fact, is some $\varepsilon > 0$ such that if we just increase our input a little bit, the output value becomes bigger than what we allow with our $\varepsilon$, and the function is therefore not continuous: It is discontinuous!




**(a)** The function $f$ **(b)** Example of $\varepsilon$

**Figure 3:** A function with a "break"

Definition 2.2 allows us to know, with certainty, that a gap in the graph, like in the figure above, cannot happen for a continuous function. No matter how far we zoom in or whether or not the jumps are so small that the computer cannot display them. Without looking at the graph of a function, or anything like that, if we know whether or not Definition 2.2 is satisfied, we can determine whether or not the graph has gaps.

A bit of intuition: Examine Figure 2, and think about the following: We know that for any $\varepsilon$ we can find some number $\delta$ such that Definition 2.2 is satisfied, but could we use a number $\alpha > 0$ which is 'smaller' than $\delta$ as our tolerance in the definition? Would the tolerance in the output still be smaller than $\varepsilon$? The reason why I want you to look at Figure 2 is that if $\delta$ were smaller, a 'smaller' interval is admitted on the $x$-axis, and, for sure, all the function's output values for numbers in this smaller interval are contained in the interval prescribed by $\varepsilon$ on the $y$-axis. So yes, a smaller number would still satisfy Definition 2.2. A more mathematically rigorous explanation is that if some number $a$ is smaller than $b$, then

$$x < a \implies x < b.$$

So if $\alpha$ is smaller than $\delta$, then

$$|x - x_0| < \alpha \implies |x - x_0| < \delta.$$

Now, time for a really mean question: Look at the graph in Figure 3 and examine $x = 0$ (which is where the $y$-axis intersects with the $x$-axis) and think about this:

**Is $f$ continuous at $x = 0$?**

Now, you do not have the distinct recipe for $f$, and I have just told you that simply looking at the graph is not always enough, but I am not trying to trick you; it is as simple as it looks. The function $f$ really is continuous at $x = 0$. "But wait," I hear you cry, "you just said $f$ wasn't continuous"! Fear not dear reader, I shall explain. When we were discussing the function $f$ in Figure 3 in the context of Definition 2.2, we were talking about a specific point, namely $x_0$. This is because continuity is a 'local' property, which means that we delimitate the examination of the function at any other point than e.g. $x_0$.

There are, of course, plenty of functions that follow the definition of continuity at every point at which they are defined, and thus the local property of continuity becomes global. The nuances of these local versus global properties are cause for much discussion and headaches in more advanced math but ultimately fall outside the scope of this article. A lot of functions taught up to, and including, 9th grade are continuous globally, and I am going to list some of them to exemplify; I encourage you to draw their graphs yourself by using GeoGebra, for example.

$p(x) = x$

$d(x) = x^2 + x + 1$

$q(x) = \sin(x)$

$k(x) = |x|$

As you may have noticed, these functions are not called $f$; there are 4 new letters here to deal with! This is to emphasize that the 'name' we give these functions is not that important and to show that if you have several in play at once, you have to name them differently to avoid confusion. This should also serve to remind you that when we say $f$, we are not talking about a 'specific' function, but just 'some' function. In the same way, the variable $x$ is not a 'specific' number, but just 'some' number.

Now that you have hopefully understood Definition 2.2 in some detail, I am going to state and prove a theorem using it.

**Theorem 2.3.** *Sum of continuous functions*
Let $f$ and $g$ be functions such that:

$$f : \mathbb{R} \to \mathbb{R},$$
$$g : \mathbb{R} \to \mathbb{R}.$$

Let both $f$ and $g$ be continuous at $x_0 \in \mathbb{R}$. Then the function

$$h : \mathbb{R} \to \mathbb{R},\ h(x) = g(x) + f(x)$$

is also continuous at $x_0$.

**Proof.** The theorem tells us that when two continuous functions are added together, the resulting function is also continuous.
The theorem states that $h$ is continuous at $x_0$. Consequently, for this to be true, we must be able to find some $\delta$ for every $\varepsilon > 0$ per Definition 2.2. So, the first thing that we can assume in this proof is that some random $\varepsilon > 0$ is given to us arbitrarily. This notion of arbitrariness suggests that while we do not know what value $\varepsilon$ takes, we should be able to find a fitting $\delta$ for any value of $\varepsilon$ that some random person might choose for us.

We first look at what is written after the 'implies' arrow in Definition 2.2, write it in terms of $h$ and rewrite:

$$\begin{aligned} |h(x) - h(x_0)| &= |(g(x) + f(x)) - (g(x_0) + f(x_0))| && \text{Insert definition of } h, \\ &= |g(x) + f(x) - g(x_0) - f(x_0)| && \text{Cancel the parentheses,} \\ &= |(f(x) - f(x_0)) + (g(x) - g(x_0))| && \text{Reorganize.} \end{aligned}$$

Now we are going to 'split the absolute value'. Essentially, if $a$ and $b$ are real numbers, the following is always true:

$$|a + b| \leq |a| + |b|.$$

This might not be obvious to you, but it is a well-proven mathematical theorem (the triangle inequality), which has a few different versions and pertaining proofs. If you want to convince yourself that this is true, you can check *Funktioner af en og flere variable*, p. 256, or the webpage *The Triangle Inequality* at *Proofwiki.org*. The triangle inequality allows us to state the following:

$$\begin{aligned} |h(x) - h(x_0)| &= |(f(x) - f(x_0)) + (g(x) - g(x_0))| && \text{What we just found,} && (2) \\ &\leq |(f(x) - f(x_0))| + |(g(x) - g(x_0))| && \text{Using the triangle inequality.} && (3) \end{aligned}$$

Next, we can start using our assumptions: Namely that both $f$ and $g$ are continuous at $x_0$. This means, by Definition 2.2, that no matter what value $\varepsilon$ takes, there exists some number $\delta > 0$ such that

$$|x - x_0| < \delta \implies |(f(x) - f(x_0))| < \varepsilon.$$

This is true no matter what $\varepsilon$ is based on our assumption that $f$ is continuous in $x_0$. Moreover, since this is true for every value of $\varepsilon$, we deduce that there has to be some number $\delta_1 > 0$ such that

$$|x - x_0| < \delta_1 \implies |(f(x) - f(x_0))| < \frac{\varepsilon}{2}. \tag{4}$$

Using the same reasoning, there has to be some number $\delta_2 > 0$ such that

$$|x - x_0| < \delta_2 \implies |(g(x) - g(x_0))| < \frac{\varepsilon}{2}. \tag{5}$$

We now have three facts to work with: Equation (3) is always true and the other two inequalities are true for some numbers $\delta_1$ and $\delta_2$. Moreover, we know from earlier that for whatever $\delta$ corresponds to any given $\varepsilon$, we may use a number smaller than that $\delta$, and Definition 2.2 would still be true. So let $\delta$ be the smallest of $\delta_1$ and $\delta_2$. Then, for this $\delta$, inequalities (4) and (5) would both be true and we can thus write the following:

$$|x - x_0| < \delta \implies |h(x) - h(x_0)| \leq |(f(x) - f(x_0))| + |(g(x) - g(x_0))| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, we have now proven the existence of a $\delta$ corresponding to the unknown $\varepsilon$ given at the start of the proof, and we have therefore proven that $h$ is continuous $\square$.

You might ask: Why bother? Well, there is a similar theorem telling you that the product of two continuous functions is also continuous. It is in fact easy to check that the following two functions are continuous everywhere:

$$\begin{aligned} &i : \mathbb{R} \to \mathbb{R},\ i(x) = x \\ &c : \mathbb{R} \to \mathbb{R},\ c(x) = c;\ \text{ a constant function for any number } c \in \mathbb{R} \end{aligned}$$

Do it! This, together with the theorems about sums and products of continuous functions, is enough to show that *every* polynomial (like $p(x) = 5x^4 - 23x^3 + 7x^2 - 9x + 8$) is continuous; no need to check continuity for every individual polynomial! Can you see why?

In most math textbooks, and also *Funktioner af en og flere variable*, the proof of Theorem 2.3 and the intuition behind Definition 2.2 would not be as exhaustively or pedagogically explained as what I have attempted to do here. They would be mixed up with a bunch of other definitions and theorems, and a lot of the required knowledge and intuition required for understanding proofs of more complicated theorems would come in the form of exercises. It is deceptively easy to read a theorem and proof and think that you got it, at least until you have to use that theorem to solve an exercise or explain that proof to someone else. Therefore, I suggest that you take a look at a few exercises straight from *Funktioner af en og flere variable*, pp. 349-350.

**Final Comments**

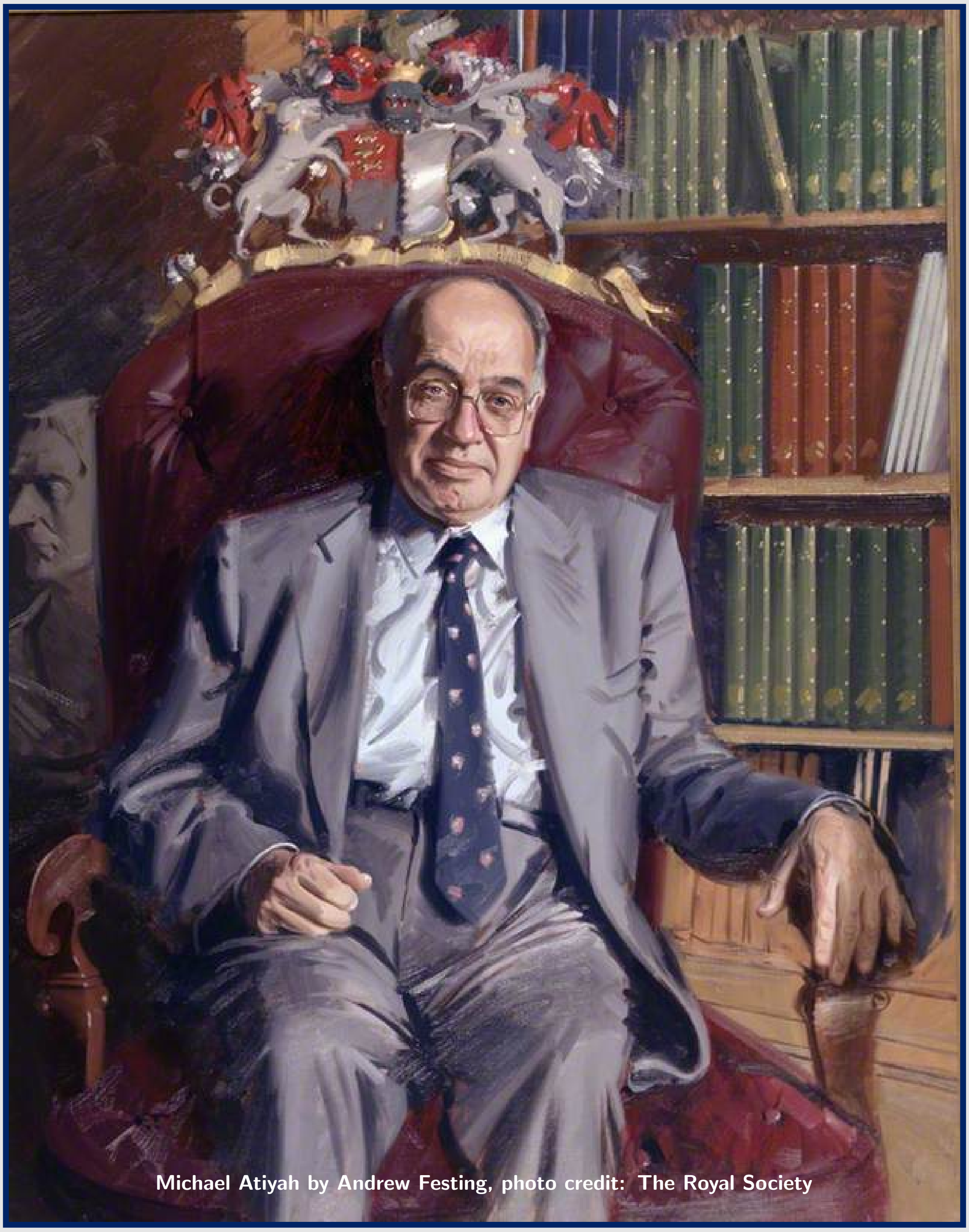
Different universities certainly have different methods as to how higher levels of mathematics courses are taught. At AAU, where I am studying, there is an even split for each course: Half the time is spent in the lecture hall and the other half doing exercises like the ones suggested above.

The reason why I bring this up is that there is a common image of how mathematics is done; that there is this anti-social, genius-loner, locked away in some room somewhere, who just keeps staring at the thing until they figure it out. This image is simply false. Mathematics, when learned and taught right, is a group activity: Doing exercises, explaining proofs, and poking holes in each others' understanding of proofs is how you learn math. I think this is true for most other disciplines as well: We learn by examining things from different angles and putting them in the context of everything else. It just so happens that in mathematics, it is easier to pinpoint exactly where one's understanding begins and ends.

No one can jam an understanding into your head, and, consequently, you have to work for it. However, talking about it is part of that work, and everyone who is good at math got there by doing a lot of work. My point is that you do not have to be scared of math, a lot of the people whom I know through my studies were not math geniuses when they started; this is exceptionally rare. They have just put in the work required to obtain their degree, and I respect them a lot for it. My hope is that this article has demystified what 'Real Math' is, and how it is done.

My message is not that mathematics is easy, and that every person you pull off the street gets it and can do high-level and abstract proofs; far from it. Mathematics is a very difficult discipline initially, and you should be wary of people in pop culture trying to woo you with easy graphs and quick-and-too-smooth math. The communication of most mathematics is very difficult, especially to a general audience; it is simply a hard body of knowledge to grasp.

I hope that these pages have given you an inkling of what happens when one studies mathematics at the university, and I hope that you think that they have been worth your time. I can be contacted at mathlearn@protonmail.com.



Michael Atiyah by Andrew Festing, photo credit: The Royal Society



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